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Financial Operative Measures

AAT Level I

BMS - Business Mathematics and Statistics

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JMC JAYASEKERA MANAGEMENT CENTRE (PVT) LTD Estd. 1972

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Chapter 4

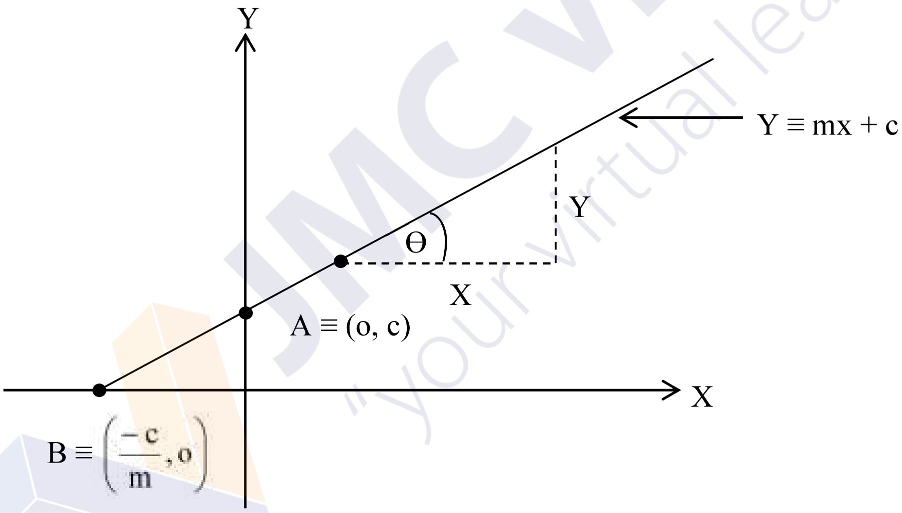
GRAPHICAL PRESENTATIONS

- Linear Functions
- Making Equations of Linear Functions
- Linear Models of Business
- Quadratic Functions
- Quadratic Model in Business

LINEAR FUNCTIONS

Any linear relationship between two variables X and Y can be described as a linear function. Graph of a linear function is a straight line to draw a graph of a linear function, we need only any two points that lie on a line.

Standard format of a straight line is Y = mx + c, where 'm' is a slope of a line and 'c' is Y - intercept. This can be shown in a diagram given below.



Slope (Gradient)

Tangent of the angle that a straight line makes positive direction of X-axis

$$m = \text{Tan}(\Theta)$$

Y-intercept

Place where a straight line cuts the Y-axis. It can be evaluate by using following method.

$Y = mx + c$

* when $x = 0$

$Y = m(0) + c$

$Y = c$

At point A ≡ (o, c) straight line cuts the Y-axis

X-intercept

Place where a straight line cuts the X-axis. It can be evaluate by using following method.

$Y = mx + c$

* when $Y = 0$

$0 = mx + c$

$-c = mx$

$\frac{-c}{m} = x$

At point B ≡ ($\frac{-c}{m}$, o) straight line cuts the X-axis

Exercise 01

Evaluate, (i) slope (ii) Y-intercept (iii) X-intercept of following straight lines.

(i)	$Y = 3x+5$	(v)	$5x-7Y = 35$
(ii)	$-Y = 2x+7$	(vi)	$x+2Y-9 = 0$
(iii)	$6Y = 12x-4$	(vii)	$Y = 4x$
(iv)	$3x+4Y = 12$	(viii)	$8x-9Y = 0$

MAKING EQUATIONS OF STRAIGHT LINES

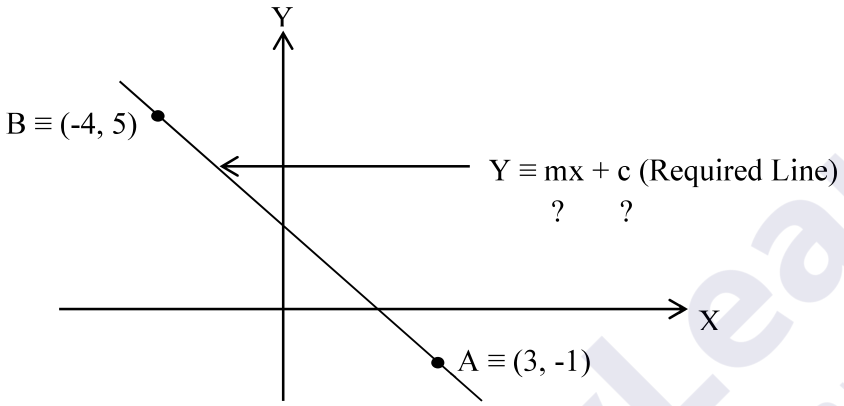
If given that coordinates of any two points there is a method of making equation of straight line.

Worked Examples

<u>Example ①</u>

Find the equation of a straight line passes through two points (3, -1) and (-4, 5)

Solution



* If it passes through a point A ≡ (3, -1) when x = 3, Y = 1

$Y \equiv mx + c$
$-1 = m(3)+c$
$-1 = 3m+c \longrightarrow$ ①

* If it passes through a point B ≡ (-4, 5) when x = -4, Y = 5

$Y \equiv mx + c$
$5 = m(-4)+c$
$5 = -4m+c \longrightarrow$ ②

By solving pair of equations ① and ②, We can evaluate 'm' and 'c'

$3m+c = -1 \longrightarrow$ ①
$-4m+c = 5 \longrightarrow$ ②
① - ②
$(3m+c) - (-4m+c) = -1 - 5$
$3m + \cancel{c} + 4m - \cancel{c} = -6$
$3m + 4m = -6$
$7m = -6$
$m = \frac{-6}{7}$ (slope)

Substitute m = $\frac{-6}{7}$ to equation $\longrightarrow$ ①

3m + c = -1

$3\left(\frac{-6}{7}\right) + c \quad = -1$

$\frac{-18}{7} + c \quad = -1$

$c \quad = -1 + \frac{18}{7}$

$c \quad = \frac{-7}{7} + \frac{18}{7}$

$c \quad = \frac{11}{7}$ (Y-intercept)

Equation of a required line Y = mx + c

$Y = \frac{-6}{7}x + \frac{11}{7}$ (Standard Format)

* Other Formats

7Y = -6x +11

7Y +6x = 11

7Y +6x -11 = 0

Exercise 01

(a) Find the equation of a straight line passes through two points (1, -2) and (3, 4)

(b) Find the equation of a straight line passes through two points (11, 6) and (10, -5)

Exercise 02

Following market research into the demand for a particular product, it has been established that there is a linear relationship between sales quantity (q) and selling price (p) when,

* P = 10 Rs. per unit sales are expected to be 240 units.
* P = 20 Rs. per unit sales are expected to be 160 units.

Evaluate equation of a demand function by assuming a linear relationship between price and demand.

LINEAR MODELS IN BUSINESS

Total Cost Function (T_C)

Total cost of manufacturing firm for a given period is usually considered as a contribution of two types of costs namely (i) Variable Cost (ii) Fixed Cost

Variable Cost

Variable cost that changes directly according to the number of units produced.

Examples

* Supply of Raw Material
* Supply of Labour

The total variable cost can be expressed as the product of the variable cost per unit and the number of units produced.

Fixed Cost

Fixed cost does not depend with the number of items manufacturing.

Examples

* Building Rent
* Maintenance Cost
* Insurance Premium

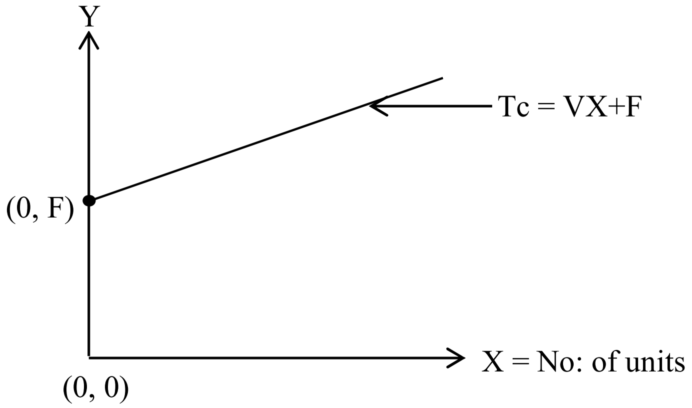
We can make a Linear Model for Total Cost Function as follows.

Total Coat = Variable Cost + Fixed Cost

$$T_c = VX + FC$$
$$\uparrow \quad \uparrow\uparrow \quad \uparrow$$
$$Y = mx + c$$

* Tc = Total cost
* V = Variable cost per unit
* X = Number of units produced
* FC = Fixed cost

<u>Graph</u>



Linear model of a Total Cost Function is a graph with

- Positive Slope
- Positive Y-intercept

Total Revenue Function (T_R)

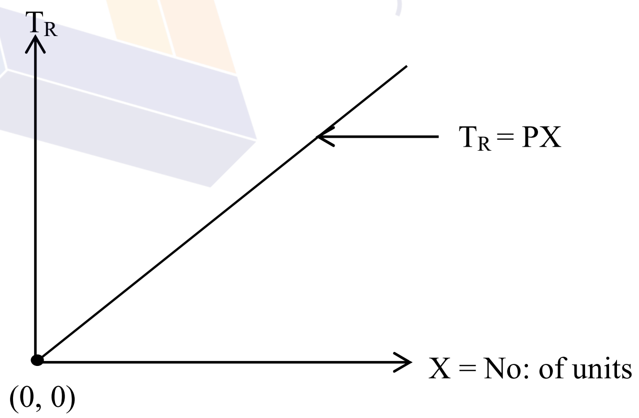
Revenue is income from sales. It refers to the total amount of money received from sales of given quantity of goods or services provided to customers during a certain time period. We can make a linear model for total revenue in a following way.

Total Revenue = (Selling Price per units) × (No: of units)

$$T_R = PX$$
$$\uparrow \quad \uparrow\uparrow$$
$$Y = mx$$

- T_R = Total Revenue
- P = Selling Price per unit
- X = Number of units sold

<u>Graph</u>



Linear model of a Total Revenue Function is a straight line with

- Positive Slope
- Passes through point (0, 0)

Total Profit Function (T_P)

For a business venture, the gross profit can be considered as the difference between the revenue obtainable from the sale of a number of products and cost involved in their production. Therefore it is difference between Revenue and cost. We can make a linear model for total profit in a following way.

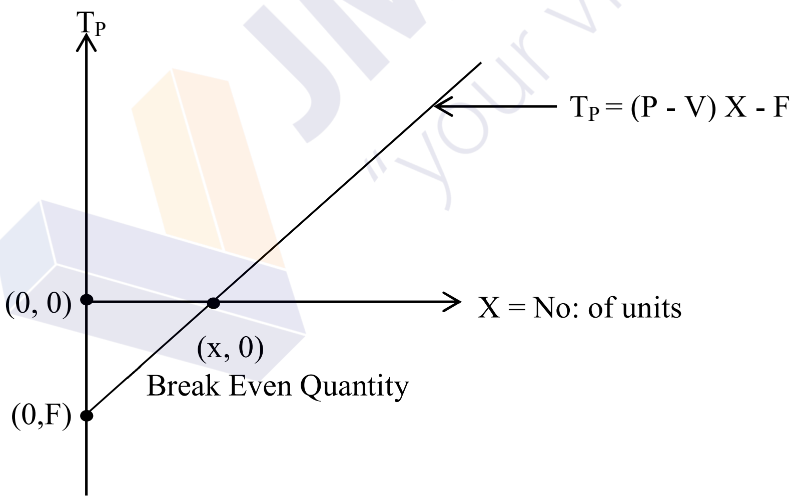
Total Profit = (Total Revenue) - (Total Cost)

$T_P = T_R - T_C$
$T_P = PX - (VX+F)$
$T_P = PX - VX - F$
$T_P = (P - V) X - F$

$$T_P = (P - V)\,X - F$$
$$Y = m\,x + c$$

- T_P = Total Profit
- P = Selling Price per unit
- V = Variable cost per unit
- X = Number of units produced and sold
- F = Fixed cost

Graph

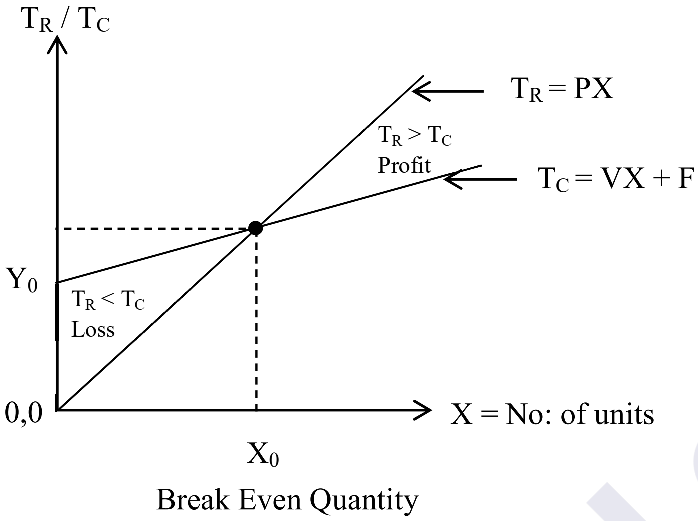


Linear model of a Total Profit Function is a straight line with

- Negative Y-intercept

Break Even Analysis

Suppose that total cost and revenue functions of a particular product is $T_C = VX + F$ and $T_R = PX$ respectively. If we draw graphs of cost and revenue functions on the same set of coordinate axes, two lines will be intersected at a particular point. The point of intersection is known as a break even point. It has shown in a diagram given below.



Therefore for Break Even Points

$$T_R = T_C$$
$$T_R - T_C = 0$$
$$T_P = 0$$

Worked Examples

Example ①

A company produces and markets a mechanical tool which has a variable cost of Rs. 25 unit. A mechanical tool sold at Rs. 50 per unit and fixed cost of the company Rs. 100,000 per month.

(a) Create suitable models for total cost, revenue and profit.
(b) Evaluate Break Even Point
(c) Evaluate total profit when (i) 6000 units (ii) 1000 units produced and sold.
(d) Draw graphs of cost and revenue functions on the same set of coordinate axes.

Solution

(a) (i) Total Cost

$T_C = VX + F$
$V = Rs. 25$
$F = Rs. 100{,}000$
$\underline{T_C = 25X + 100{,}000}$

(ii) Total Revenue

$T_R = PX$
$P = Rs. 50$
$\underline{T_R = 50X}$

(iii) Total Profit

$T_P = T_R - T_C$
$T_P = 50X - (25X + 100{,}000)$
$T_P = 50X - 25X - 100{,}000$
$\underline{T_P = 25X - 100{,}000}$

(b) Break Even Points

For break even points

$T_R = T_C$
$50X = 25X + 100{,}000$
$50X - 25X = 100{,}000$
$25X = 100{,}000$
$X = \frac{100{,}000}{25}$
$X = 4000$ units

Sub: $X = 4000$ to T_R
$T_R = 50X$
$T_R = 50\,(4000)$
$T_R = 200{,}000$

Break Even Point = (4000, 200,000)

(c) Evaluate Profit

$T_P = 25X - 100,000$

* when X = 6000
  T_P = 25 (6000) - 100,000
  T_P = 150,000 - 100,000
  T_P = Rs. 50,000

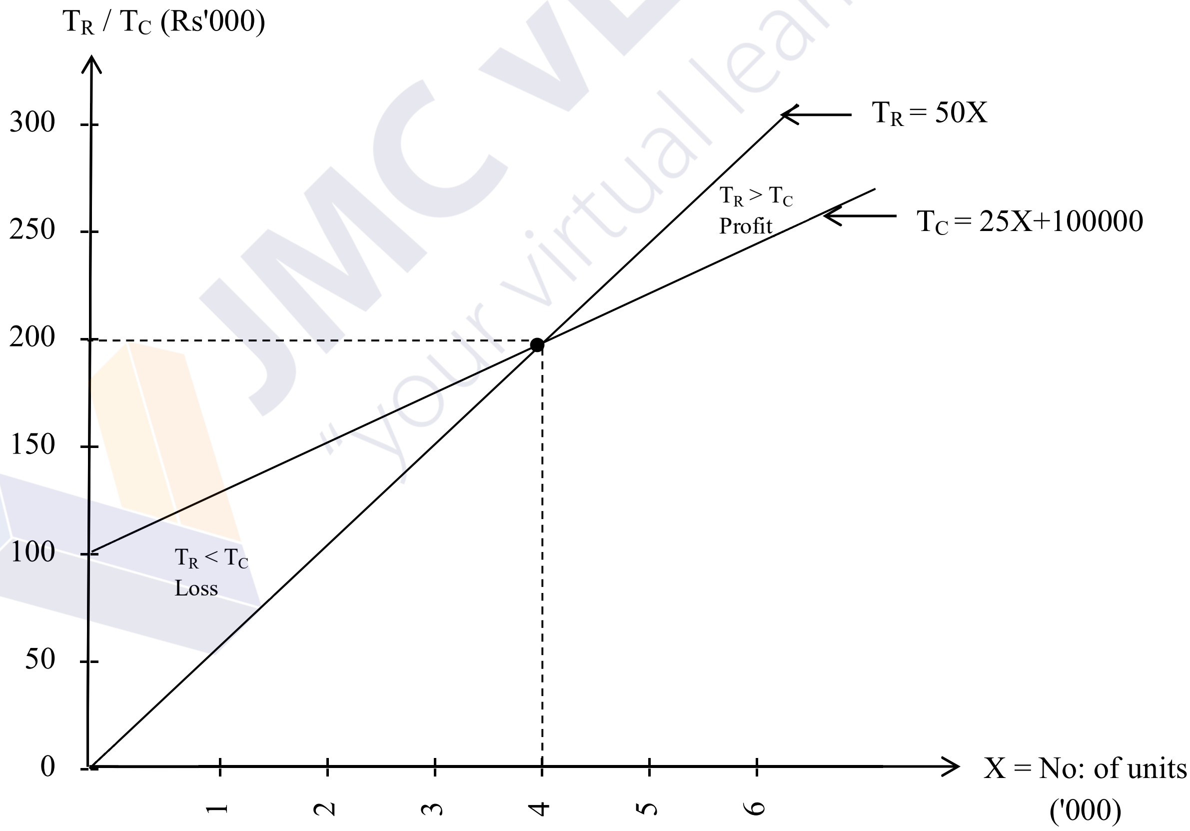
* when X = 1000
  T_P = 25 (1000) - 100,000
  T_P = (75,000) Loss

(d) Graph

$T_C = 25X + 100,000$ $T_R = 50X$

X	T_C
0	100000
5000	225000

X	T_R
0	0
6000	300000



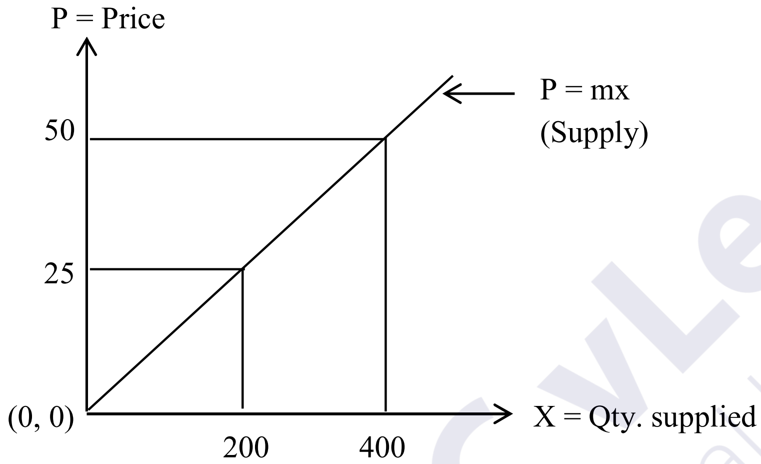
Exercise 01

A firm's fired cost is Rs. 60,000. It costs the firm Rs. 140 to produce one unit. Selling price ±r unit is Rs. 155.

(a) Evaluate Total Cost, Revenue and Profit Functions
(b) Calculate Break Even Point
(c) Draw Graphs of all 3 functions on the same set of coordinate axes.

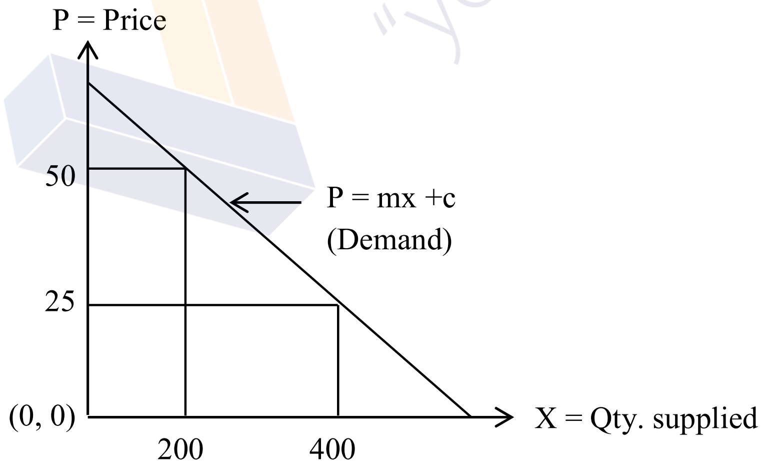
Linear Supply Function

There is a relationship in between quantity supplied and price per unit. When price increases, quantity supplied is also increases. Therefore linear supply function is a straight line with a positive slope.



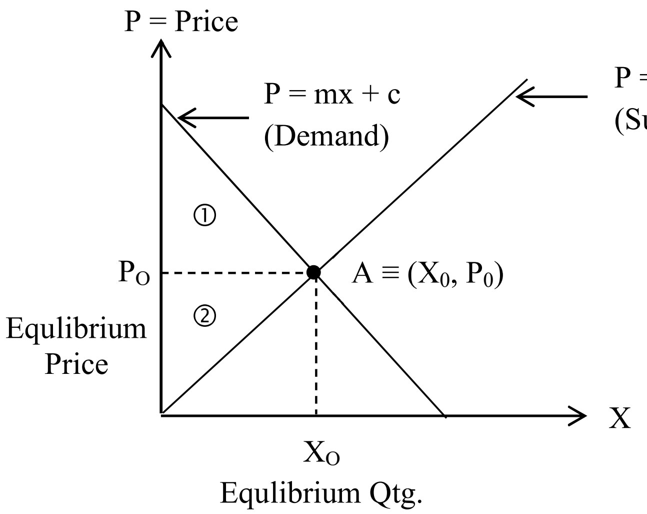
Linear Demand Function

There is a relationship in between quantity demand and price per unit. When price increases, quantity demand decreases. Therefore linear demand function is a straight line with a negative slope.



Market Equlibrium Point

If we draw graphs of cost and revenue function on the same set of coordinate axes, two lines will be intersected at a particular point what is known as market equlibrium point.



Therefore for Market Equlibruim Points

Supply = Demand

According to the diagram given above

- Consumer's Surplus = Area of Triangle ①
- Producer's Surplus = Area of Triangle ②

Worked Examples

Example ①

The demand and supply functions for a commodity is D: $P = 22 - 2q$
S: $P = 2q + 10$

Where q quantity in units and P is price per unit in rupees

(a) Evaluate Market Equlibrium Point

(b) Draw sketch diagrams of supply and demand function on the same set of coordinate axes.

(c) By using a diagram in (b) evaluate

  (i) Consumer's Surplus

  (ii) Producer's surplus

Solution

(a) Market Equlibruim Point

Given that

D: P = 22 - 2q ⟶ ①

S: P = 2q + 10 ⟶ ②

For Market Equlibruim Point

Demand = Supply

① = ②

22 - 2q = 2q + 10

12 = 4q

q = 3 units

(Equlibrium Quantity)

Sub: q = 3 to ①

P = 22 - 2q

P = 22 - 2(3)

P = 22 - 6

P = Rs. 16

Equlibrium Point ≡ (3, 16)

(b) Sketch Diagram

Supply

P = 2q + 10

↑ ↑↑ ↑

Y = mx + c

Supply function cuts the Y-axis at point (0, 10)

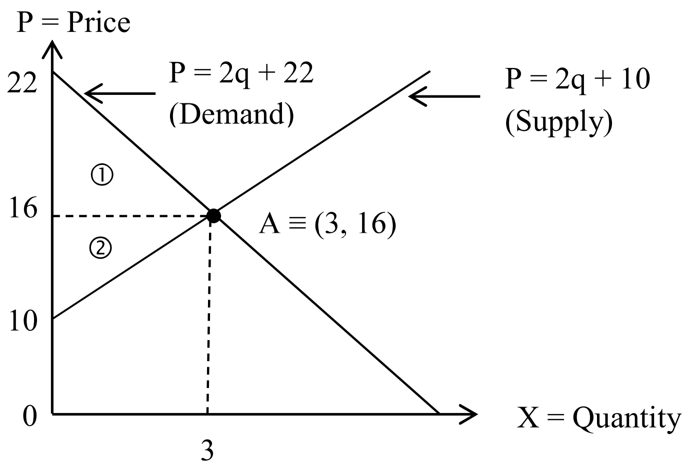
Demand

P = 2q - 22

↑ ↑↑ ↑

Y = mx + c

Supply function cuts the Y-axis at point (0, 22)



(c) (i) Consumer's Surplus

C_S = Area of Triangle ①

$C_S = \frac{1}{2} \times 3 \times (22 - 16)$

$C_S = \frac{1}{2} \times 3 \times 6$

C_S = Rs. 9

(ii) Producer's surplus

P_S = Area of Triangle ②

$P_S = \frac{1}{2} \times 3 \times (16 - 10)$

$P_S = \frac{1}{2} \times 3 \times 6$

P_S = Rs. 9

Exercise 01

Suppose that the supply function for a product is S = 500P and the demand function is d=-1000P+120000, where P is price in dollars, S is quantity supplied, D is quantity demand.

(a) Evaluate Market Equlibrium Point
(b) Draw sketch diagrams of supply and demand function on the same axes.
(c) Evaluate Consumer's and Producer's surplus.

QUADRATIC FUNCTIONS

Standard format of a quadratic function is $Y = ax^2 + bx + c$ where a, b and c are real numbers and $a \neq 0$

Examples

*	$Y = 3x^2 - 2x + 8$	(a = 3	b = -2	c = 8)
*	$Y = -x^2 + 8$	(a = -1	b = 8	c = 0)
*	$Y = x^2 + 7$	(a = 1	b = 0	c = 7)
*	$Y = 5x^2$	(a = 5	b = 0	c = 0)

<u>Sketch Graph of a Quadratic Function</u>

Let $Y = ax^2 + bx + c$

<u>Type of a Turning Point</u>

* a > 0 (Positive) - Minimum Point
* a < 0 (Negative) - Maximum Point

<u>Cordinates of Turning Point</u>

$$X = \frac{-b}{2a}$$

$$Y = c - \left(\frac{b^2}{4a}\right)$$

$$\text{Turning Point} \equiv \left\{\left(\frac{-b}{2a}\right), c - \left(\frac{b^2}{4a}\right)\right\}$$

<u>Y-intercept</u>

$Y = ax^2 + bx + c$
* When x = 0

$Y = a(0)^2 + b(0) + c$
$Y = c$
At point (0,c) Graph cuts the Y-axis.

Worked Examples

<u>Example ①</u>

Draw sketch diagrams of following functions
(a) $Y = X^2 - 10x + 21$
(b) $Y = X^2 + 6x$
(c) $Y = X^2 + 6x - 9$
(d) $Y = X^2 - 2x + 5$

Solution

(a) $Y = X^2 - 10x + 21$

$a = 1 \quad b = -10 \quad c = 21$

* <u>Type of a Turning Point</u>

$a = 1 > 0$ (Positive) - Minimum Point

* <u>Coordinates</u>

$X = \frac{-b}{2a}$ $\qquad$ $Y = c - \left(\frac{b^2}{4a}\right)$

$X = \frac{-(-10)}{2(1)}$ $\qquad$ $Y = 21 - \frac{(-10)^2}{4(+1)}$

$X = \frac{10}{2}$ $\qquad$ $Y = 21 - 25$

$X = 5$ $\qquad$ $Y = -4$

Minimum Point ≡ (5, -4)

* <u>Y-intercept</u>

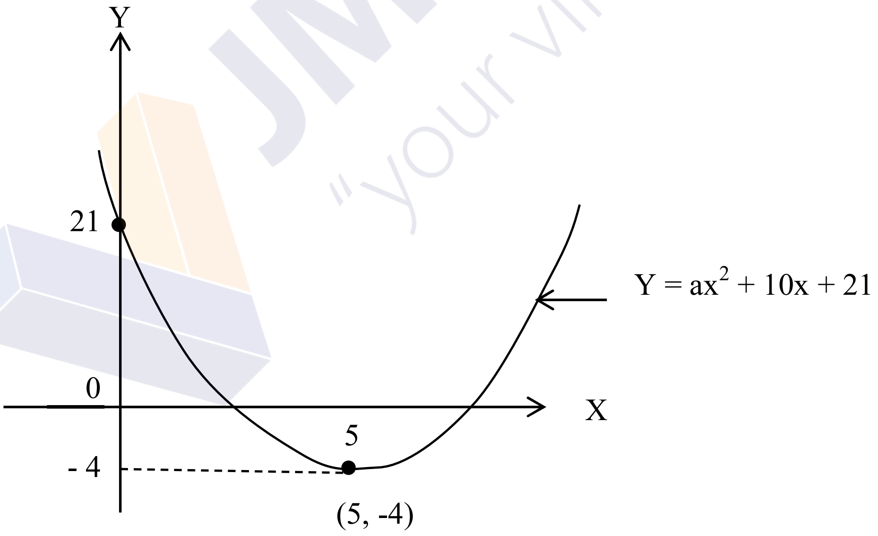
$Y = ax^2 + 10x + 21$

* When $x = 0$

$Y = (0)^2 - 10(0) + 21$

$Y = 21$

Graph cuts the Y-axis at point (0,21)



Minimum Point

(b) $Y = -X^2 + 6x$

$a = -1 \quad b = 6 \quad c = 0$

* <u>Type of a Turning Point</u>

$a = -1 < 0$ (Negative) - Maximum Point

* <u>Coordinates</u>

$X = \frac{-b}{2a}$

$X = \frac{-6}{2(-1)}$

$X = \frac{-6}{-2}$

$X = 3$

$Y = \left(\frac{b^2}{4a}\right)$

$Y = 0 - \frac{(6)^2}{4(-1)}$

$Y = 0 - \left(\frac{36}{-4}\right)$

$Y = 0 + 9$

$Y = 9$

Maximum Point ≡ (3,9)

* <u>Y-intercept</u>

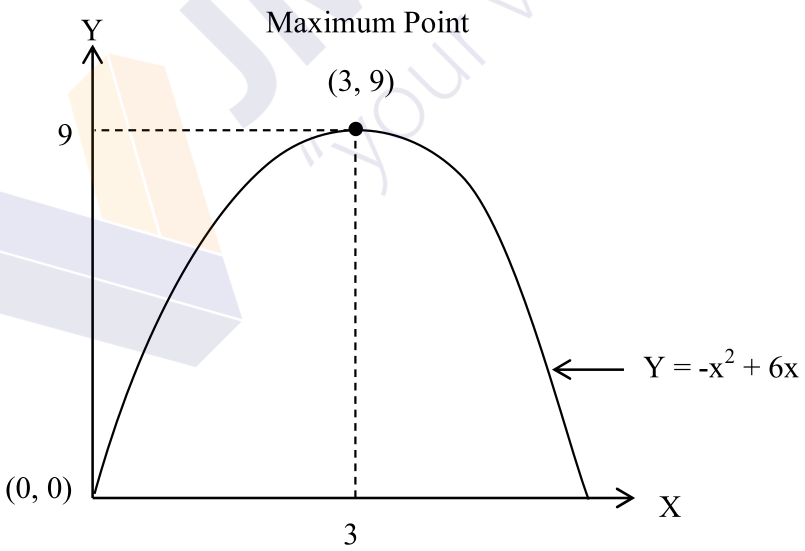
$Y = -x^2 + 6x$

* When $x = 0$

$Y = -(0)^2 + 6(0)$

$Y = 0$

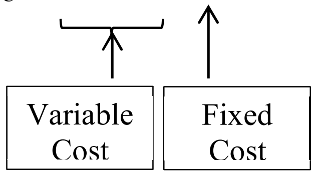
Graph passes through point (0, 0)



QUADRATIC MODELS IN BUSINESS

Total Cost Function (T_C)

$T_C = ax^2 + bx + c$



* a > 0 (Positive)

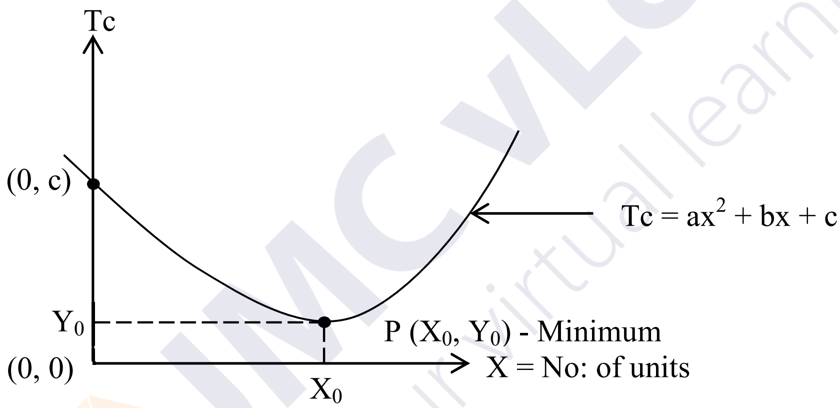
When a > o, graph has a minimum point Total cost of a company should be minimum.

* c > 0

Y-intercept represent the fixed cost of a company.

We can make a quadratic model for total cost as follows.
$T_C = ax^2 + bx + c$

Graph



Therefore quadratic model of a Total Cost Function is a graph with

* Minimum point

* Positive Y-intercept

Total Revenue Function (T_R)

$T_R = ax^2 + bx + c$

* a < 0 (Negative)

When a < o, graph has a maximum point Total Revenue of a company should be maximum.

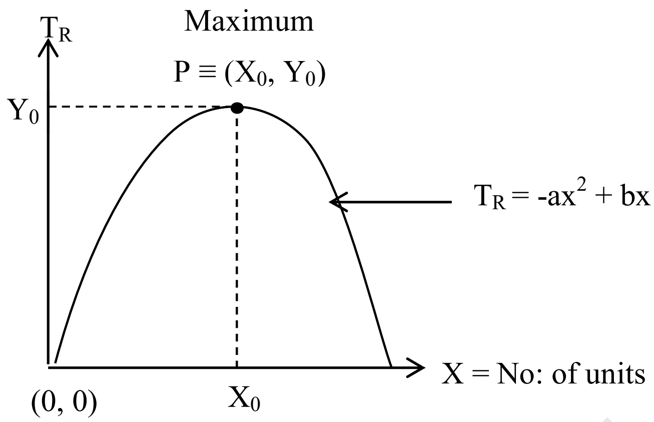
* c = 0 (Zero)

Total revenue is multiple of selling price per unit and quantity sold. When quantity sold is equal to zero, there is no revenue to company. Therefore graph passes through point (0, 0)

We can make a quadratic model for total cost as follows.

$T_R = -ax^2 + bx$

<u>Graph</u>



Therefore quadratic model of a Total Revenue Function is a graph with

* Maximum Point
* Passes through point (0, 0)

Total Profit Function (T_P)

$T_P = ax^2 + bx + c$

* <u>a < 0 (Negative)</u>

  When a < o, graph has a maximum point. Total Profit of a company should be maximum.

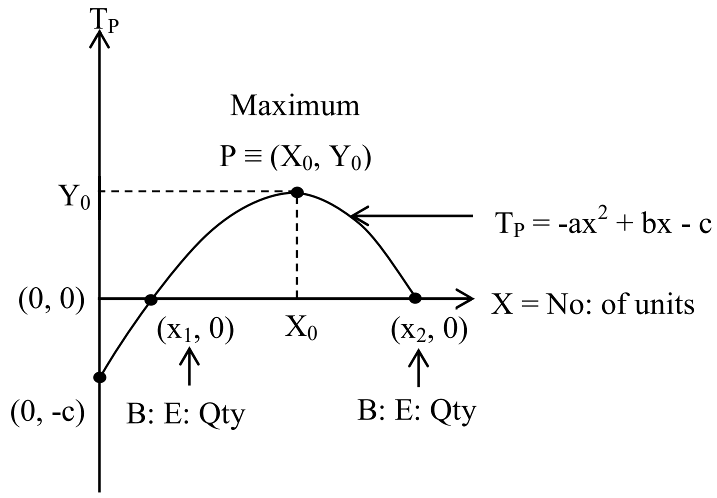
* <u>c < 0 (Negative)</u>

  Profit is difference between revenue and cost. All revenue functions passes through point (0, 0) and all cost functions has a positive Y-intercept. When take T_R - T_C, profit function 0btained negative Y-intercept.

  We can make a quadratic model for total profit as follows.

  $T_P = -ax^2 + bx - c$

Graph



Therefore quadratic model of a Total Profit Function is a graph with

- * Maximum Point
- * Negative Y-intercept

Worked Examples

Example ①

A manufacturing firm producing items has a fixed has a fixed cost of Rs. 12,500 per day and variable cost of Rs. 50 per unit when x items are produced and sold the demand function is x=800 - 4P, where P is Selling Price per unit.

(a) Make suitable models for total cost, revenue and profit
(b) Evaluate break-even point
(c) Draw graphs of cost and revenue functions on the same set of cordinate axes
(d) By using the above graph find the production level that yields a profit for the company.

Solution

(a) (i) Total Cost

$T_C = VX + F$
$V = Rs. 50$
$F = Rs. 12,500$

$\underline{T_C = 50X + 12,500}$ (Linear)

(ii) Total Revenue

$T_R = PX$
$X = 800 - 4P$
$4P = 800 - X$
$P = \frac{800 - X}{4}$
$P = \frac{800}{4} - \frac{X}{4}$
$P = 200 - 0.25X$
$T_R = (200 - 0.25X)$
$T_R = 200X - 0.25X^2$ (Quadratic)

(iii) Total Profit

$T_P = T_R - T_C$
$T_P = (200X - 0.25X^2) - (50X + 12{,}500)$
$T_P = 200x - 0.25X^2 - 50X - 12{,}500$
$T_P = -0.25X^2 + 150X - 12{,}500$

(b) Break Even Point

$T_P = -0.25X^2 + 150X - 12{,}500$
For B: E: Points $T_P = 0$
$-0.25X^2 + 150X - 12{,}500$
$X = 100 \quad X = 500$

* When X = 100

$T_C = 50X + 12{,}500$
$T_C = 50(100) + 12{,}500$
$T_C = 5{,}000 + 12{,}500$
$T_C = 17{,}500$

BEP I ≡ (100, 17500)

* When X = 500

$T_C = 50X + 12{,}500$
$T_C = 50(500) + 12{,}500$
$T_C = 25{,}000 + 12{,}500$
$T_C = 37{,}500$

BEP I ≡ (500, 37500)

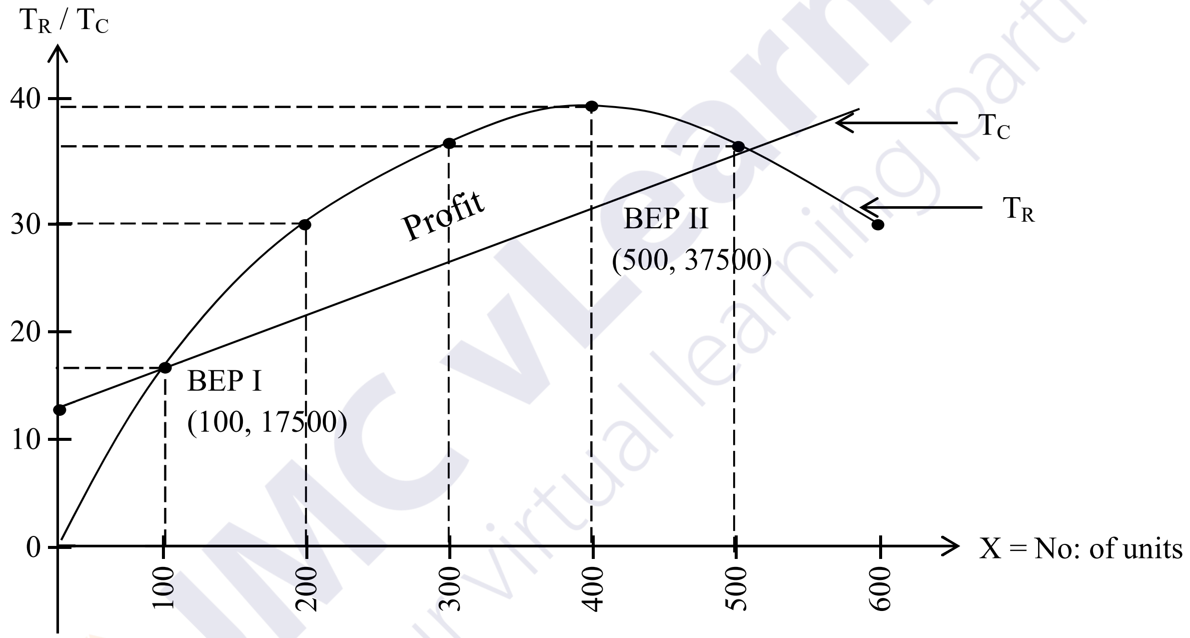
(c) Graph

$T_R = -0.25X + 200X$

X	0	100	200	300	400	500	600
T_R	0	17500	30000	37500	40000	37500	30000

$T_C = 50X + 12,500$

X	0	600
T_C	12500	42500



(d) Production level that yields a profit for the company 0 < X < 500

Example ②

A company produces and sells a single product has weekly fixed costs of Rs. 900000 and variable costs of Rs. $(10000q + 1000q^2)$ where 'q' is weekly output. The weekly demand / price function for this product is P = 120000 - 1000q. Where 'P' is price per unit.

(a) Calculate break even weekly quantities

(b) Calculate quantity and price at profit maximum point

Solution

(a) Total Cost Function

Total Cost = Variable Cost + Fixed Cost

TC = $(10000q + 1000q^2) + 900000$

TC = $1000q^2 + 10000q + 900000$

Total Revenue Function

T_R = (Selling Price per unit) + (No: of units sold)

T_R = Pq

T_R = (120000 - 1000q) q (Demand Function)

T_R = $120000q - 1000q^2$

T_R = $-1000q^2 + 120000q$

Total Profit Function

$T_P = T_R - T_C$

$T_P = -1000q^2 + 120000q - (1000q^2 + 10000q + 900000)$

$T_P = -1000q^2 + 120000q - 1000q^2 - 10000q - 900000$

$T_P = -2000q^2 + 110000q - 900000$

For B:E Points $T_P = 0$

$-2000q + 110000q - 900000 = 0$

q = 10 units and q = 45 units

Break Even Quantities:

q = 10 units and q = 45 units

(b) Quantity and Price at Profit Maximum Point

$T_P = -2000q^2 + 110000q - 900000$

a = -2000 b = 110000 c = -900000

Qty for max profit = X - cordinate of max. point

$$q = \frac{-b}{2a}$$

$$q = \frac{-110000}{2(-2000)}$$

$$q = \frac{-110000}{-4000}$$

<u>q = 27.5 units</u>

$$q = \frac{10+45}{2}$$

<u>q = 27.5 units</u>

<u>Price</u>

From a demand function
p = 120000 - 1000q

* When q = 27.5

P = 120000 - 1000 (27.5)
P = 120000 - 27500
<u>P = Rs. 92500</u>

<u>Exercise 01</u>

A company produces and sells a product. Its weekly fixed costs have been estimated as Rs.7,200,000 and variable fixed of $9000q + 2q^2$, where q is the quantity produced per week. The company's capacity is about 100 per day. The demand function for this product is given by P=24000 - 3q, where 'P' is unit price and 'q' is weekly quantity sold.

(a) Calculate the Cost, Revenue and Profit in terms of q
(b) Calculate break even quantities
(c) State the number of units the company should produce every week.

Exercise 01

A manufacturing of a product has found that he can sell 70 units per day ditect to the customer if the price is Rs. 48,000. In error the price was recently advertised at Rs. 78,000 and as a result, only 40 units were sold per day. The manufacturer's fixed cost of production are Rs. 1,170,000 per day and variable cost are Rs. 9000 per unit.

(a) Compute the demand function linking price (P) to quantity demand (x), assuming it to be STRAIGHT LINE.

(b) State cost of firm

(c) Compute the unit price, which maximize profit and quantity demanded and profit generated at that price

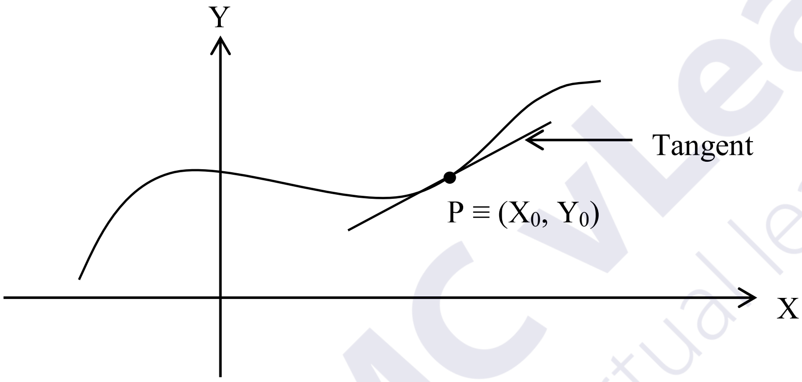
Chapter 5

DIFFERENTIAL CALCULUS

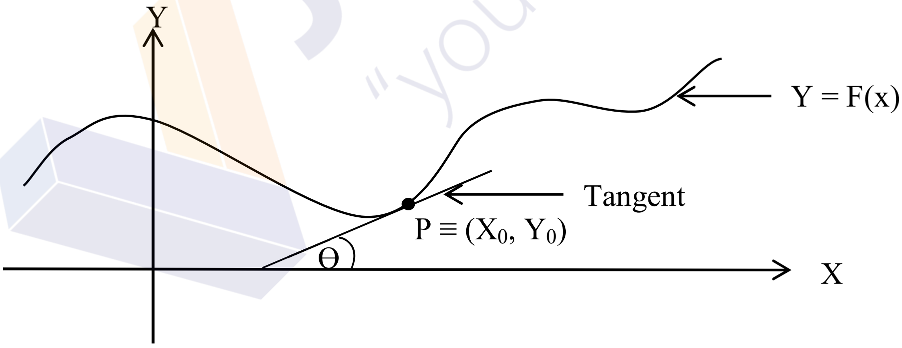
* Differential Coefficient
* Rules of Differentiation
* Turning points of a curve
* Business Applications

DIFFERENTIATION

Differentiation is a process which transforms one function into a different one. The new function is known as a deravative of theoriginal one. A derivative is a slope of a line tangent toa curve at a point. The tangent is a line which goes through only one point of a curve.



Differential Coefficient $\left(\frac{dY}{dX}\right)$



$\frac{dY}{dX}$ = Slope of a curve Y = F(x) at point $P \equiv (X_0, Y_0$

RULES OF DIFFERENTIATION

Rule Number ①

$$Y = k$$
$$\frac{dY}{dX} = 0$$

Worked Examples

Example ①

Differentiate following with the respect to variable 'x'

a) $Y = 5$

$\frac{dY}{dX} = 0$

b) $Y = -8$

$\frac{dY}{dX} = 0$

c) $Y = \frac{4}{5}$

$\frac{dY}{dX} = 0$

Exercise 01

(a) $Y = 3$ (b) $Y = -12$ (c) $Y = \frac{4}{3}$ (d) $Y = \frac{-2}{7}$

(e) $Y = 1.25$ (f) $Y = -3.38$ (g) $Y = \sqrt{2}$ (h) $Y = \frac{-1}{\sqrt{3}}$

Rule Number ②

$$Y = x^n$$
$$\frac{dY}{dX} = nx^{n-1}$$

Worked Examples

Example ①

Differentiate following with the respect to variable 'x'

a) $Y = x^5$

$\frac{dY}{dX} = 5x^4$

b) $Y = x^{-10}$

$\frac{dY}{dX} = 10x^{-11}$

c) $Y = x^{4/3}$

$\frac{dY}{dX} = \frac{4}{3}x^{1/3}$

d) $Y = \frac{1}{x^6}$

$Y = x^{-6}$

$\frac{dY}{dX} = 6x^{-7}$

e) $Y = 5\sqrt{x}$

$Y = x^{1/5}$ $\frac{1}{5} - 1 = \frac{1}{5} - \frac{5}{5} = \frac{-4}{5}$

$\frac{dY}{dX} = \frac{1}{5}x^{-4/5}$

Exercise 01

(a)	$Y = x^7$	(b)	$Y = x$	(c)	$Y = x^{-2}$	(d)	$Y = x^{-4}$
(e)	$Y = x^{9/2}$	(f)	$Y = x^{-2/3}$	(g)	$Y = \frac{1}{x^8}$	(h)	$Y = \frac{1}{x^2}$
(i)	$Y = \sqrt{x}$	(j)	$Y = \frac{-1}{3\sqrt{x}}$				

Rule Number ③

$$Y = kx^n$$

$$\frac{dY}{dX} = knx^{n-1}$$

Worked Examples

Example ①

Differentiate following with the respect to variable ‘x’

a) $Y = 9x^5$

$\frac{dY}{dX} = 9\,(4x^3)$

$\frac{dY}{dX} = 36x^3$

b) $Y = -10x^{-3}$

$\frac{dY}{dX} = 10\,(-3x^{-4})$

$\frac{dY}{dX} = 30x^{-4}$

c) $Y = \frac{2}{5}x^6$

$\frac{dY}{dX} = \frac{2}{5}\,(6x^5)$

$\frac{dY}{dX} = \frac{12}{5}x^5$

d) $Y = \frac{-3}{8}x^{-4}$

$\frac{dY}{dX} = \frac{-3}{8}\,(-4x^5)$

$\frac{dY}{dX} = \frac{3}{2}x^{-5}$

e) $Y = \frac{5}{x^7}$

$Y = 5x^{-7}$

$\frac{dY}{dX} = 5(-7x^{-8})$

$\frac{dY}{dX} = -35x^{-8}$

f) $Y = \frac{4}{7x^2}$

$Y = \frac{4}{7}\left(\frac{1}{x^2}\right)$

$Y = \frac{4}{7}x^{-2}$

$\frac{dY}{dX} = \frac{4}{7}\ (-2x^{-3})$

$\frac{dY}{dX} = \frac{-8}{7}x^{-3}$

Exercise 01

(a) $Y = 8x^3$ (b) $Y = -10x^5$ (c) $Y = 12x^{-4}$ (d) $Y = -7x^{-5}$

(e) $Y = \frac{4}{3}x^6$ (f) $Y = \frac{-2}{5}x^{-7}$ (g) $Y = \frac{4}{x}$ (h) $Y = \frac{-9}{x^{10}}$

(i) $Y = \frac{11}{3x^2}$ (j) $Y = \frac{-9}{2x^3}$ (k) $Y = \sqrt[3]{x}$ (l) $Y = \frac{-2}{\sqrt{x}}$

Rule Number ④

$$Y = F(x) \pm G(x)$$
$$\frac{dY}{dX} = \frac{dF(x)}{dX} \pm \frac{dG(x)}{dX}$$

Worked Examples

Example ①

Differentiate following with the respect to variable 'x'

a) $Y = ax + 13$

$\frac{dY}{dX} = 9\ (1) + 0$

$\frac{dY}{dX} = 9$

b) $Y = 5x^2 - 12x + 19$

$\frac{dY}{dX} = 5(2x^{-1}) - 12(1) + 0$

$\frac{dY}{dX} = 10x - 12$

c) $Y = x^3 + 5x^2 - 19x + 10$

$\frac{dY}{dX} = 3x^2 + 5(2x) - 19(1) + 0$

$\frac{dY}{dX} = 3x^2 + 10x - 19$

d) $Y = \frac{2}{8}x^5 - \frac{3}{8}x^4 - \frac{1}{12}x^3$

$\frac{dY}{dX} = \frac{2}{9}(5x^4) - \frac{3}{8}(4x^3) - \frac{1}{12}(3x^2)$

$\frac{dY}{dX} = \frac{10}{9}x^4 - \frac{12}{8}x^3 - \frac{3}{12}x^2$

e) $Y = x^6 - \frac{5}{x^3}$

$Y = x^6 - 5x^{-3}$

$\frac{dY}{dX} = 6x^5 - 5(-3x^{-4})$

$\frac{dY}{dX} = 6x^5 + 15x^{-4}$

Exercise 01

(a) $Y = 8x - 9$

(b) $Y = x^2 + 11x - 20$

(c) $Y = 5x^2 + 8x + 11$

(d) $Y = x^3 - 20x^2 - 17x + 6$

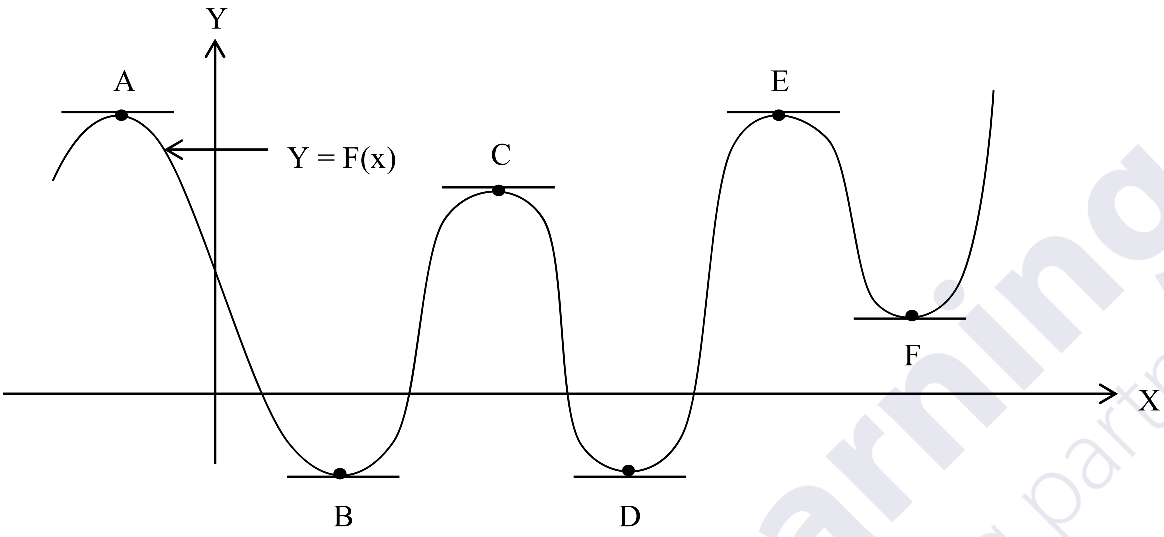
(e) $Y = 3x^4 - 9x^3 + 12x^2$

(f) $Y = \frac{4}{9}x^2 - \frac{11}{5}x + \frac{1}{4}$

(g) $Y = 4x^3 + \frac{4}{x^2}$

(h) $Y = \frac{10}{x^8} - \frac{15}{x^7} - \frac{5}{x^6}$

TURNING POINTS OF A CURVE



The above diagram shows a graph of a function Y = F(x). The point A, B, C, D, E and F are turning points. Points A, C and E are maximum points. Points B, D and F are minimum points.

At a turning point tangent to the curve is parallel to the x-axis. Therefore the slope of a tangent of a turning point is zero. Since $\frac{dY}{dX}$ measures the turning of a tangent of any point on the curve, at a turning point $\frac{dY}{dX} = 0$.

Therefore at Turning Points

$$\frac{dY}{dX} = 0$$

To confirm whether a Turning Point is Minimum or maximum

Let P ≡ (X_0, Y_0) be a turning point of a function Y = F(x).

(a) Evaluate $\frac{d^2Y}{dX^2}$ (Second Order Differential Coefficient)

(b) Find the value of $\frac{d^2Y}{dX^2}$ at a turning point.

(i) If $\frac{d^2Y}{dX^2} > 0$ (Positive) - Minimum Point

(ii) If $\frac{d^2Y}{dX^2} < 0$ Negative) - Maximum Point

(iii) If $\frac{d^2Y}{dX^2} = 0$ (Zero) - Inflextion Point

Worked Examples

Example ①

Find the turning points of a curve $Y = 2x^3 - 3x^2 - 36x + 30$ Check whether are they minimum. Draw a sketch diagram of the above function.

Solution

Evaluate Turning Points

$Y = 2x^3 - 3x^2 - 36x + 30$

$\frac{dY}{dX} = 2(3x^2) - 3(2x) - 36(1) + 0$

$\frac{dY}{dX} = 6x^2 - 6x - 36$

For turning points $\frac{dY}{dX} = 0$

$6x^2 - 6x - 36 = 0$

∗ a = 6 ∗ b = -6 ∗ c = -36

∗ x = 3, ∗ x = -2

$Y = 2x^3 - 3x^2 - 36x + 30$

∗ When x = 3

$Y = 2(3)^3 - 3(3)^2 - 36(3) + 30$

$Y = 54 - 27 - 108 + 30$

$Y = -51$

(3, -51) is a turning point.

$Y = 2x^3 - 3x^2 - 36x + 30$

* When x = -2

$Y = 2(-2)^3 - 3(-2)^2 - 36(-2) + 30$
$Y = -16 - 12 + 72 + 30$
$Y = 74$

(-2, 74) is a turning point.

To check whether are they minimum or maximum

$\frac{dY}{dX} = 6x^2 - 6x - 36$
$\frac{d^2Y}{dX^2} = 6(x) - 6(1) - 0$
$\frac{d^2Y}{dX^2} = 12x - 6$

* At point (3, -21)

$\frac{d^2Y}{dX^2} = 12x - 6$

* When x = 3

$\frac{d^2Y}{dX^2} = 12(3) - 6$
$\frac{d^2Y}{dX^2} = 30$ (Positive)

(3, -51) is a Minimum Point.

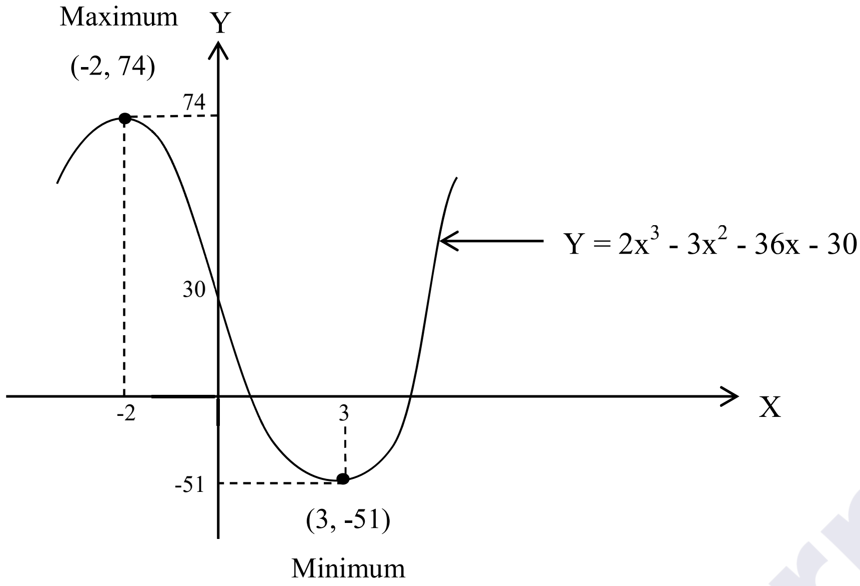
* At point (-2, 74)

$\frac{d^2Y}{dX^2} = 12x - 6$

* When x = 2

$\frac{d^2Y}{dX^2} = 12(-2) - 6$
$\frac{d^2Y}{dX^2} = -24 -6 = -30$ (Negative)

(-2, 74) is a Maximum Point.



Exercise 01

Find the turning points of a curve $Y = 2x^3 - 3x^2 - 36x - 30$. Check whether are they minimum or maximum. Draw a sketch diagram of a function.

BUSINESS APPLICATIONS OF DIFFERENTIATION

MARGINAL COST FUNCTION (M_C)

Let the equation of a cost carve at point can be defined as $\frac{dT_C}{dx}$, according to the definition of differential coefficient.

Marginal cost is an approximate change in cost resulting from additional unit of output. Since $\frac{dT_C}{dx}$ measure the rate of change in cost with the respect to quantity. Then we can regard $\frac{dT_C}{dx}$ as measure of marginal cost.

Therefore,

$$M_C = \frac{dT_C}{dx}$$

In the same manner

$$M_R = \frac{dT_R}{dx}$$

$$M_P = \frac{dT_P}{dx}$$

Where

T_C	= Total Cost
T_R	= Total Revenue
T_P	= Total Profit
M_C	= Margin Cost
M_R	= Margin Revenue
M_P	= Margin Profit
x	= No: of units produced and sold

Worked Example

Example ①

The Total Cost Function of a firm is given by $T_C = 0.001x^3 - 0.24x^2 + 40x + 1400$. Where x is number of units produced. Evaluate,

(i) Marginal Cost Function
(ii) Marginal Cost when 60 units produced

Solution

(i) <u>Marginal Cost Function</u>

$T_C = 0.001x^3 - 0.24x^2 + 40x + 1400$

$M_C = \frac{dT_R}{dx} = 0.001\ (3x^2) - 0.24\ (2x) + 40\ (1)\ 40$

$\underline{M_C = 0.003x2 - 0.48x + 40}$

(ii) <u>Marginal Cost when 60 units produced</u>

$M_C = 0.003x^2 - 0.48x + 40$

* When x = 60

$M_C = 0.003\ (60)^2 - 0.48\ (60) + 40$

$\underline{M_C = Rs.\ 22}$

<u>Exercise 01</u>

Total Revenue Function of a particular product is given by $T_R = 800x - \frac{x^2}{4}$, where x is number of units produced and sold. Evaluate,

(i) Marginal Revenue Function

(ii) Marginal Revenue for 100 units

<u>Exercise 02</u>

The daily profit function of a firm is given by $T_R = 90x - \frac{x^2}{20} - 6000$. where x is number of units produced and sold. Evaluate,

(i) Marginal Profit Function

(ii) Marginal Profit for 200 units

Worked Example

<u>Example ①</u>

Total cost function and demand function for a product is given by $T_C = q^2 - 60q + 3000$ and q=300 - p respectively. Where q is number of units and p is price per unit.

(i) Total Revenue Function

(ii) Total Profit Function

(iii) Level of output which maximize profit

(iv) Maximum Profit

(v) Selling price per unit when profit maximized.

Solution

(i) Total Revenue Function

$T_R = Pq$

From a demand function

$q = 300 - p$

$p = 300 - q$

$T_R = (300 - q)\, q$

$T_R = 300q - q^2$

$T_R = -q^2 + 300q$ (Quod)

(ii) Total Profit Function

$T_P = T_R - T_C$

$T_P = -q_2 + 300q - (q^2 - 60q + 3000)$

$T_P = q_2 + 300q - q_2 + 60q - 3000$

$T_P = 2q^2 + 360q - 3000$ (Quod)

(iii) Level of output which maximize profit

$T_P = -2q^2 + 360q - 3000$

$\frac{dT_P}{dq} = -2(2q) + 360\,(1) - 0$

$\frac{dT_P}{dq} = -4q + 360$

For max/min, points $\frac{dT_P}{dq} = 0$

$-4q + 360 = 0$

$4q = 360$

$q = 90$

$\frac{dT_P}{dq} = -4q + 360$

$\frac{d^2T_P}{dq^2} = -4\,(1) + 0$

$\frac{d^2T_P}{dq} = -4$

At q = 90, there is a maximum point

Level of output which maximize Profit is = 90

(iv) Maximum Profit

$T_P = -2q^2 + 360q - 3000$

* when $q = 90$

$T_P = 2(90)^2 + 360(90) - 3000$

$T_P = \text{Rs. } 13{,}200$

(v) Selling price per unit when profit maximized

$q = 300 - p$

* when $q = 90$

$90 = 300 - p$

$T_P = \text{Rs. } 210$

Exercise 01

Total Revenue Function for a particular product is given by $T_R = 500q - 2q^2$, where q is number of units sold. Find,

(i) Level of output which maximize revenue

(ii) Maximum Revenue

(iii) Draw a sketch diagram of revenue function

Exercise 02

Total Cost Function for a product is given by $T_C = q^2 - 300q + 50000$, where q is number of units produced. Find,

(i) Level of output which maximize cost

(ii) Minimum cost

(iii) Draw a sketch diagram of cost function